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Electromagnetic field propagation in neutral isotropic plasmas

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Abstract. Interested in electromagnetic waves inside plasmas with known dielectric properties, we prove that Courant-Hilbert fields of the focus wave mode type can propagate in a steady-state, isotropic, neutral plasma which is in fact a dielectric with the refractive index $n^2(\omega) = 1 - \omega_p^2/\omega^2$ where ω_p is the plasma frequency. We also prove that neutral, isotropic plasmas with memory behave for harmonic plane waves as a time reversal mirror when the memory function is a decreasing exponential.

PACS. 42.25.Bs Wave propagation, transmission and absorption – 52.35.Hr Electromagnetic waves (e.g., electron-cyclotron, Whistler, Bernstein, upper hybrid, lower hybrid) – 52.27.Aj Single-component, electron-positive-ion plasmas

1 Introduction

Harmonic plane waves are generally used to investigate the propagation of electromagnetic waves in plasmas [1–3] because their dielectric properties are known when they are excited by harmonic fields and also because they make calculations easier with the idea that radiowave propagation in cosmic plasmas is correctly described by plane waves. This assumption leads for instance in a steady-state, isotropic plasma to look for the solutions of the partial differential equation satisfied by the electric field $\mathbf{E}$ ([3], Chap. 2)

$$\begin{aligned}&\boldsymbol{\Delta}\mathbf{E} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) + \omega^2 c^{-2}(\mathbf{D} - i4\pi\omega^{-1}\mathbf{j}) = 0,\\ &\mathbf{D} = \varepsilon\mathbf{E}, \qquad \mathbf{j} = \sigma\mathbf{E}\end{aligned} \tag{1}$$

in which ε, σ are the permittivity and the conductivity of the plasma so that equation (1) becomes

$$\boldsymbol{\Delta}\mathbf{E} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) + \omega^2 c^{-2}\,\varepsilon'\mathbf{E} = 0, \quad \varepsilon' = \varepsilon - i4\pi\sigma/\omega. \tag{1a}$$

This equation simplifies for transverse waves since in this case $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$.

Our objective is different and we investigate in a steady-state, isotropic, neutral plasma with known dielectric properties, the propagation of nondiffractive electromagnetic waves, as they exist in free space, able to transmit information at large distances and presenting a more realistic approximation of the experimental situation as plane waves.

It has been proved a long time ago by Courant-Hilbert [4] that the wave equation in free space $\{c = 1,\ \mathbf{x} = (x, y, z)\}$

$$(\Delta - \partial_t^2)\psi(\mathbf{x}, t) = 0, \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \tag{2}$$

has nondiffractive solutions (undistorted progressing waves in their terminology) of the type

$$\psi(\mathbf{x}, t) = A(\mathbf{x}, t)\, F[\Omega(\mathbf{x}, t)] \tag{3}$$

in which F is an arbitrary function (with continuous derivatives), Ω a solution of the characteristic equation linked to the wave equation (2)

$$(\partial_x\Omega)^2 + (\partial_y\Omega)^2 + (\partial_z\Omega)^2 - (\partial_t\Omega)^2 = 0 \tag{4}$$

while the amplitude A does not depend on F. In addition to plane and spherical waves, the best known non diffractive waves are the so-called focus wave modes of angular frequency ω in which $F(\Omega) = \exp(i\omega\Omega)$, rediscovered simultaneously but independently by Brittingham [5] and Kiselev [6] in their quest to get less diffractive beams as the conventional laser beams, solutions of the paraxial wave equation.

So, a natural question is whether electromagnetic waves with similar properties can still propagate in a plasma with properties not too far from free space such as a steady-state, isotropic, neutral plasma. In such a medium, Maxwell's equations take the form

$$\begin{aligned}&\boldsymbol{\nabla}\wedge\mathbf{E}(\mathbf{x}, t) + \partial_t\mathbf{H}(\mathbf{x}, t) = 0,\\ &\boldsymbol{\nabla}\wedge\mathbf{H}(\mathbf{x}, t) - \partial_t\mathbf{E}(\mathbf{x}, t) = 4\pi\mathbf{j}(\mathbf{x}, t)\end{aligned} \tag{5a}$$

with in addition the divergence equations, the first one being obtained from the linearized Poisson equation

$$\boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{x}, t) = -4\pi e\,\rho(\mathbf{x}, t), \quad \boldsymbol{\nabla}\cdot\mathbf{H}(\mathbf{x}, t) = 0 \tag{5b}$$

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while the current $\mathbf{j}(\mathbf{x},t)$ induced in the plasma by the electric field satisfies the Ohm's law

$$\partial_t \mathbf{j}(\mathbf{x},t) = (ne^2/m)\mathbf{E}(\mathbf{x},t), \quad \omega_p^2 = 4\pi ne^2/m \tag{6}$$

and ω_p is the plasma frequency.

In these expressions, e, m are the charge and the mass of electrons, ne the equilibrium charge density of ions and electrons and ρ the charge density.

We investigate afterwards a problem a bit different but still in the same mood of discussing electromagnetic wave propagation in plasmas with known dielectric properties by considering what happens in plasmas endowed with memory. Some works [7,8] in which the memory effects come from non-Markovian dielectric fluctuations have been recently published but here we assign a different origin to the memory effects by supposing that the current induced in the plasma depends not only on the electric field at the time t but also on its past history as the displacement current in a Maxwell-Hopkinson dielectric [9,10] so that we have to deal with a generalized Ohm's law.

This paper in which the propagation is supposed in the z-direction to avoid cumbersome notations is organized as follows: Section 2 is devoted to propagation in a 2D-plasma of TE, TM, Courant-Hilbert fields when electromagnetic wave polarization can be neglected making calculations very simple. The same problem is analyzed in Section 3 for Courant-Hilbert waves with arbitrary polarization propagating in a steady-state, isotropic, neutral 3D-plasma and we get a simple generalization of the free space focus wave modes. Section 4 is concerned with electromagnetic plane waves inside a plasma with memory described by an induced current of the Maxwell-Hopkinson type: a particular attention is given to exponential memory functions supplying a time reversal mirror behaviour of the plasma. Conclusive comments are given in Section 5.

2 TE, TM Courant-Hilbert fields

Assuming that the electromagnetic field polarization can be neglected, we may suppose that one of the two transverse coordinates x, y: say y, does not play any role. Then $E_x = E_z = H_y = 0$ for TE waves implying $j_x = j_z = 0$ and the Maxwell equations (5a) reduce to

$$\begin{aligned} &\partial_t H_x = \partial_z E_y, \qquad \partial_t H_z = -\partial_x E_y, \\ &\partial_z H_x - \partial_x H_z = 4\pi j_y + \partial_t E_y. \end{aligned} \tag{7}$$

Performing the time derivative of the third equation (7) and using the first two equations together with equations (6), a simple calculation shows that the y-component E_y is solution of the Klein-Gordon equation

$$(\partial_x^2 + \partial_z^2 - \partial_t^2 - \omega_p^2)E_y = 0 \tag{8}$$

from which the other two components may be obtained.

Similarly $H_x = H_z = E_y = 0 = j_y$ for TM waves and equations (5a) reduce to

$$\begin{aligned} &\partial_z H_y = -(4\pi j_x + \partial_t E_x), \qquad \partial_x H_y = 4\pi j_z + \partial_t E_z, \\ &\partial_z E_x - \partial_x E_z + \partial_t H_y = 0 \end{aligned} \tag{9}$$

and obtaining the Klein-Gordon equation satisfied by H_y is a bit less simple than for TE fields. Using (6), we get from the first two equations (9)

$$\partial_t \partial_z^2 H_y = -(\partial_t^2 + \omega_p^2)\partial_z E_x, \quad \partial_t \partial_x^2 H_y = (\partial_t^2 + \omega_p^2)\partial_x E_z. \tag{9a}$$

Summing these two relations and taking into account the third equation (9) give finally

$$(\partial_x^2 + \partial_z^2 - \partial_t^2 - \omega_p^2)\partial_t H_y = 0. \tag{10}$$

Consequently, to get the TE, TM electromagnetic waves propagating in a steady-state, isotropic neutral plasma, we have to solve th 2D-Klein-Gordon equation

$$(\partial_x^2 + \partial_z^2 - \partial_t^2 - \omega_p^2)\psi(x,z,t) = 0 \tag{11}$$

with the corresponding characteristic equation

$$(\partial_x \Omega^2) + (\partial_z \Omega^2) - (\partial_t \Omega^2) = 0 \tag{12}$$

satisfied by the linear phases $t \pm z$ but also by the Gaussian phases

$$\Omega(x,z,t) = t - z - x^2(t+z+ia)^{-1}, \quad i = \sqrt{-1}, \quad a > 0 \tag{12a}$$

used in conventional 2D-focus wave modes (also called arrow wave modes [11]) to generalize the 2D-Gaussian beams.

With the phase (12a), the Courant-Hilbert solutions of exponential type are

$$\psi(x,z,t) = A(z,t)\exp\left[i\omega\left\{t - z - x^2(t+z+ia)^{-1}\right\}\right] \tag{13}$$

and substituting (13) into (11), we get in Appendix A the following expression of $A(z,t)$

$$A(z,t) = A(t+z+ia)^{-1/2}\exp\left[-i\omega_p^2(t+z+ia)/4\omega\right] \tag{13a}$$

(A is an arbitrary constant) reducing to the amplitude of 2D-focus wave modes [11] for $\omega_p = 0$.

We note A_e, A_h the constant A for TE, TM fields respectively and ψ_e, ψ_h the corresponding Courant-Hilbert fields. Then, using (7), (9) it is easy to get the non-null components of the electromagnetic field

$$\begin{aligned} &E_y = \partial_t \psi_e, \qquad H_x = \partial_z \psi_e, \qquad H_z = -\partial_x \psi_e \\ &H_y = (\partial_t^2 + \omega_p^2)\psi_h, \quad E_x = -\partial_t \partial_z \psi_h, \quad E_z = \partial_t \partial_x \psi_h \end{aligned} \tag{14}$$

and it is easily checked that these fields satisfy the divergence equations (5b).

It is stated in the introduction that the amplitude $A(\mathbf{x},t)$ of the Courant-Hilbert solutions of the wave equation (2) does not depend on the function F which may be arbitrary, provided that continuous derivatives of F exist. But A is compelled to satisfy an overdetermined system of differential equations [4], rather curiously fulfilled in free space, which is not the case here due to the presence of the ω_p^2 term in equation (11). So, F must be an exponential function since all its derivatives being proportional to F make sure that A will not depend on F. Consequently, all the Courant-Hilbert fields propagating in a plasma are of the focus wave modes type with harmonic plane waves as a particular case. In addition, there is no guarantee that for any phase of the characteristic equation (12), the differential equations for A supply fair solutions.

3 Courant-Hilbert electromagnetic waves

As well-known, only two components that we take here as E_y and H_y are needed to describe the full electromagnetic field since the other components are supplied by Maxwell's equations. Then, using the notation $b^2 = \partial_t^2 + \omega_p^2$ to make calculations less cumbersome and taking the time derivative of four among the six curl equations (5a) give the relations

$$\partial_y \partial_t E_z - \partial_z \partial_t E_y = -\partial_t^2 H_x, \ \partial_x \partial_t E_y - \partial_y \partial_t E_x = -\partial_t^2 H_z \quad (15a)$$

$$\partial_y \partial_t H_z - \partial_z \partial_t H_y = b^2 E_x, \quad \partial_x \partial_t H_y - \partial_y \partial_t H_x = b^2 E_z. \quad (15b)$$

Substituting (15b) into (15a) and vice versa, we get the components $H_{x,z}$, $E_{x,z}$ in terms of H_y and E_y

$$(\partial_y^2 - b^2)\partial_t^2 H_x = \partial_x \partial_y \partial_t^2 H_y - b^2 \partial_z \partial_t E_y$$
$$(\partial_y^2 - b^2)\partial_t^2 H_z = \partial_z \partial_y \partial_t^2 H_y + b^2 \partial_x \partial_t E_y \quad (16a)$$

and

$$(\partial_y^2 - b^2)\partial_t E_x = \partial_z \partial_t^2 H_y + \partial_x \partial_y \partial_t E_y$$
$$(\partial_y^2 - b^2)\partial_t E_z = -\partial_x \partial_t^2 H_y + \partial_z \partial_y \partial_t E_y. \quad (16b)$$

Now we get from the last two curl equations

$$(\partial_y^2 - b^2)\partial_z \partial_t^2 H_x - (\partial_y^2 - b^2)\partial_x \partial_t^2 H_z = b^2 (\partial_y^2 - b^2)\partial_t E_y \quad (17a)$$

$$(\partial_y^2 - b^2)\partial_z \partial_t E_x - (\partial_y^2 - b^2)\partial_x \partial_t E_z = -(\partial_y^2 - b^2)\partial_t^2 H_y. \quad (17b)$$

Substituting (16a) into (17a), (16b) into (17b) and using the definition of the b^2 operator give the partial differential equations satisfied by E_y and H_y

$$(\Delta - \partial_t^2 - \omega_p^2)\partial_t^2 H_y(\mathbf{x}, t) = 0,$$
$$(\Delta - \partial_t^2 - \omega_p^2)\partial_t E_y(\mathbf{x}, t) = 0. \quad (18)$$

So finally, we have to solve the 3D-Klein-Gordon equation

$$(\Delta - \partial_t^2 - \omega_p^2)\Psi(\mathbf{x}, t) = 0 \quad (19)$$

with the characteristic equation (4). The solution of equation (4) generalizing (12a) and also used in [5,6] is

$$\Omega(\mathbf{x}, t) = t - z - r^2(t + z + ia)^{-1}, \quad r^2 = x^2 + y^2 \quad (20)$$

so that the Courant-Hilbert solutions of the focus wave mode type have the form

$$\Psi(\mathbf{x}, t) = A(z, t) \exp\left[i\omega\left\{t - z - r^2(t + z + ia)^{-1}\right\}\right] \quad (21)$$

and substituting (21) into (19) we get in equation (A.12) of Appendix A with the arbitrary constant A

$$A(z, t) = A(t + z + ia)^{-1} \exp\left[-i\omega_p^2(t + z + ia)/4\omega\right] \quad (22)$$

reducing to the amplitudes of focus wave modes [5,6] for $\omega_p = 0$. Note that the attenuation factor $(t + z + ia)^{-1/2}$ in (13a) becomes $(t + z + ia)^{-1}$ in (22).

With the constants A_e, A_h linked to the E_y, H_y components as in Section 2 and taking into account the relations (16a, 16b) the Courant-Hilbert electromagnetic waves propagating in a stationary, isotropic, neutral plasma have the expressions in terms of the two scalar fields Ψ_e, Ψ_h

$$\begin{aligned} E_y &= (\partial_z^2 - \partial_t^2 - \omega_p^2)\partial_t \Psi_e, & E_x &= \partial_z \partial_t \Psi_h + \partial_x \partial_y \partial_t \Psi_e, \\ E_z &= -\partial_x \partial_t \Psi_h + \partial_z \partial_y \partial_t \Psi_e, & H_y &= (\partial_z^2 - \partial_t^2 - \omega_p^2)\Psi_h, \\ H_x &= \partial_x \partial_y \Psi_h & H_z &= \partial_z \partial_y \Psi_h \\ &\quad - (\partial_t^2 + \omega_p^2)\partial_z \Psi_e, & &\quad + (\partial_t^2 + \omega_p^2)\partial_x \Psi_e. \end{aligned} \quad (23)$$

A simple calculation shows that with (23) the divergence equations (5b) are satisfied.

Remark. Instead of $r^2 = x^2 + y^2$ in the phase (20) we could use $(x \cos\theta + y \sin\theta)^2$ with an arbitrary angle θ but since for $\theta = 0$, the phase (20) has the expression (12a), the attenuation factor in the amplitude A is the same as for TE and TM waves.

4 Electromagnetic waves in a plasma with memory

Leaving aside the Courant-Hilbert fields, but still interested in waves able to propagate inside a plasma with known dielectric properties, we now investigate electromagnetic field in a neutral isotropic plasma with dielectric properties such as the current induced by the electric field has a form similar to that of the displacement current proposed a long time ago by Hopkinson [9] on a Maxwell's suggestion to explain the results of his experiments on Leyden jars.

Explicitly, as a substitute to equation (6) in the introduction, the Courant $j(\mathbf{x}, t)$ induced in the plasma satisfies now the generalized Ohm's law

$$4\pi \partial_t \mathbf{j}(\mathbf{x}, t) = \omega_p^2 \mathbf{E}(\mathbf{x}, t) + \beta^2 \int_0^\infty d\tau R(\tau)\, \mathbf{E}(\mathbf{x}, t - \tau) \quad (24)$$

in which the memory function $R(t)$, $t \geq 0$ is a monotonically decreasing function of t which is continuous for $0 \leq t < \infty$ while β is the Mawxell-Hopkinson frequency.

Then, proceeding as in Section 3, the scalar field $\Psi(\mathbf{x}, t)$ linked to the components E_y, H_y by the relations (23) is solution of the partial differential equation

$$(\Delta - \partial_t^2)\Psi(\mathbf{x}, t) = \omega_p^2 \Psi(\mathbf{x}, t) + \beta^2 \int_0^\infty d\tau R(\tau)\Psi(\mathbf{x}, t - \tau) \tag{25}$$

and to solve this equation, we introduce the Laplace transform $\psi(\mathbf{x}, s)$ of $\Psi(\mathbf{x}, t)$

$$\psi(\mathbf{x}, s) = \int_0^\infty dt \exp(-st)\Psi(\mathbf{x}, t) \tag{26}$$

so that taking the Laplace transform of equation (25) gives

$$\begin{aligned}(\Delta - s^2)\psi(\mathbf{x}, s) &= -[s\Psi(\mathbf{x}, 0) + \Psi'(\mathbf{x}, 0)] + \omega_p^2\, \psi(\mathbf{x}, s) \\ &\quad + \beta^2 \int_0^\infty d\tau R(\tau) \exp(-s\tau)\, \psi(\mathbf{x}, s) \\ &= -[s\Psi(\mathbf{x}, 0) + \Psi'(\mathbf{x}, 0)] \\ &\quad + [\omega_p^2 + \beta^2 r(s)]\psi(\mathbf{x}, s) \end{aligned} \tag{27}$$

where $r(s)$ is the Laplace transform of $R(t)$ while $\Psi(\mathbf{x}, 0)$, $\Psi'(\mathbf{x}, 0)$ are the field Ψ and its time derivative at $t = 0$.

We now look for plane wave solutions of equation (27)

$$\psi(\mathbf{x}, s) = \exp(i\mathbf{k} \cdot \mathbf{x})a(s) \tag{28}$$

and substituting (28) into (27) gives the differential equation for $a(s)$ in which $k^2 = |\mathbf{k}|^2$

$$\begin{aligned}[k^2 + s^2 + \chi^2(s)]a(s) &= [sA(0) + A'(0)], \\ \chi^2(s) &= \omega_p^2 + \beta^2 r(s)\end{aligned} \tag{29}$$

$A(0)$ is the inverse Laplace transform of $a(s)$ at $t = 0$ and $A'(0)$ its time derivative.

As an illustration, we suppose that the memory function of the dielectric has the form also suggested by Hopkinson [9] $R(t) = T^{-1} \exp(-t/T)$ with the Laplace transform $r(s) = (1 + sT)^{-1}$ [12] so that according to (29)

$$\begin{aligned}&a(s) = [sA(0) + A'(0)] \left[\omega^2 + s^2 + \beta^2(1 + sT)^{-1}\right]^{-1}, \\ &\omega^2 = k^2 + \omega_p^2.\end{aligned} \tag{30}$$

We check easily that the Abel asymptotic relations [12] are fulfilled by this expression

$$\begin{aligned}\lim_{s \Rightarrow \infty} s\, a(s) &= A(0), \\ \lim_{s \Rightarrow \infty} s[a(s) - A(0)] &= A'(0).\end{aligned} \tag{31}$$

So, for s large enough to make the last term in (30) negligible we get

$$a(s) \approx [sA(0) + A'(0)] \left[\omega^2 + s^2\right]^{-1} \tag{32}$$

with the inverse Laplace transform

$$A(t_<) = A(0) \cos(\omega t) + \omega^{-1} \sin(\omega t)A'(0) \tag{32a}$$

where the notation $t_<$ means that this expression holds valid at short times $t \ll T$ after the launch of the electromagnetic pulse.

At the opposite for large times, that is for small s such that sT is much smaller than unity, the expression (30) reduces to

$$a(s) \approx [sA(0) + A'(0)] \left[\omega_1^2 + s^2\right]^{-1}, \quad \omega_1^2 = \omega^2 + \beta^2 \tag{33}$$

whose inverse Laplace transform is

$$A(t_>) = A(0) \cos(\omega_1 t) + \omega_1^{-1} \sin(\omega_1 t)A'(0) \tag{33a}$$

the notation $t_>$ meaning that (33a) is valid for $t \gg T$.

Then, with the initial conditions $A(0) = 1$, $A'(0) = i\omega$, we have

$$\begin{aligned}&A(t_<) = \exp(-i\omega t), \\ &A(t_>) = (1 + \omega/\omega_1) \exp(-i\omega_1 t) + (1 - \omega/\omega_1) \exp(i\omega_1 t)\end{aligned} \tag{34}$$

and the dielectric plasma with memory plays, coupled with a frequency-shift converter, the role of a time reversal mirror [13,14] with the reflection coefficient $(\omega_1 - \omega)(\omega_1 + \omega)^{-1}$.

5 Discussion

This work proves that electromagnetic focus wave modes can propagate in steady-state, isotropic, neutral plasmas, In fact, these waves carry on an infinite energy and therefore are not physically realizable, no more than plane waves. But, the possibility of generating finite energy approximations of these waves, known as splash wave modes [15,16], with a non-diffracting behaviour on an important distance has been proved from a theoretical point of view, leading to practical realizations in acoustics [17]. In addition, many experiments [18,19] have justified this concept of energy directed beams. So, in this part of the Universe which may be considered as a neutral plasma, very energetic electromagnetic beams could propagate, all the more undistorted as they are more energetic, very far from their source and the recently observed gamma ray bursts [20] are perhaps a manifestation of their existence.

Electromagnetic harmonic waves propagating in Maxwell-Hopkinson, neutral, isotropic plasmas present rather unusual properties that would lead to justify the generalized Ohm's law (24) if they were confirmed by experimental observations. This challenge could be taken up by coupling a charged particle modelisation of plasmas with the laws of mechanics as was made [21] for a magnetic plasma, but in this case the generalized Ohm's law has not the form (24). And the question is: what plasmas have the properties of a Maxwell-Hopkinson dielectric?

Appendix A

To get the solutions of the Klein-Gordon equation (11) in the form (13), we first deduce from (12a) the relations in which for simplification we write $D = t + z + ia$

$$\partial_x \Omega = -2xD^{-1},\ \partial_z \Omega = -1 + x^2 D^{-2},\ \partial_t \Omega = 1 + x^2 D^{-2}$$
$$\partial_x^2 \Omega = -2D^{-1},\quad \partial_z^2 \Omega = -2x^2 D^{-3},\quad \partial_t^2 \Omega = -2x^2 D^{-3} \tag{A.1}$$

so that taking into account (12) the phase Ω satisfies the relation

$$(\partial_x \Omega)^2 + (\partial_z \Omega)^2 - (\partial_t \Omega)^2 = 0, \quad \partial_z^2 \Omega - \partial_t^2 \Omega = 0. \tag{A.2}$$

So, writing the scalar field (13) in the form $\psi = A \exp(i\Omega)$, we get:

$$\partial_x \psi = i\omega A \partial_x \Omega \exp(i\omega\Omega)$$
$$\partial_z \psi = [\partial_z A + i\omega A \partial_z \Omega] \exp(i\omega\Omega) \tag{A.3}$$

giving as second derivatives

$$\partial_x^2 \psi = \left[i\omega \partial_x^2 \Omega - \omega^2 (\partial_x \Omega)^2\right] A \exp(i\omega\Omega)$$
$$\partial_z^2 \psi = \left[\partial_z^2 A + 2i\omega \partial_z A \partial_z \Omega + i\omega A \partial_z^2 \Omega - \omega^2 A (\partial_z \Omega)^2\right] \times \exp(i\omega\Omega) \tag{A.4}$$

and we have just to change ∂_z into ∂_t in this last relation to get $\partial_t^2 \psi$. Then, substituting (A.4) augmented with $\partial_t^2 \psi$ into the Klein-Gordon equation (11) and using (A.2) give the partial differential equation satisfied by the amplitude $A(z,t)$

$$(\partial_z^2 - \partial_t^2) A + 2i\omega\, (\partial_z A \partial_z \Omega - \partial_t A \partial_t \Omega) + i\omega A \partial_x^2 \Omega - \omega_p^2 A = 0 \tag{A.5}$$

in which according to (A.1)

$$\partial_z A \partial_z \Omega - \partial_t A \partial_t \Omega = -(\partial_z A + \partial_t A) + x^2 D^{-2} (\partial_z A - \partial_t A). \tag{A.6}$$

But, since A does not depend on x, the coefficient of x^2 in (A.6) must be null implying $\partial_z A = \partial_t A$ so that the partial differential equation (A.5) becomes also using the expression (A.1) of $\partial_x^2 \Omega$

$$4i\omega \partial_z A + 2i\omega\, D^{-1} A + \omega_p^2 A = 0. \tag{A.7}$$

with the solution in which D is given its explicit expression while A is an arbitrary constant

$$A(z,t) = (t + z + ia)^{-1/2} \exp[-i\omega_p^2 (t + z + ia)/\omega]. \tag{A.8}$$

Similarly in the 3D-case, we have just to change x^2 into $x^2 + y^2$ in (A.1) with in addition the derivative $\partial_y^2 \Omega = -2D^{-1}$ while the relation (A.2) becomes

$$(\partial_x \Omega)^2 + (\partial_y \Omega)^2 + (\partial_z \Omega)^2 - (\partial_t \Omega)^2 = 0,$$
$$\partial_z^2 \Omega + \partial_y^2 \Omega - \partial_t^2 \Omega = 0. \tag{A.9}$$

Then, substituting in the Klein-Gordon equation (19) the expression (A.4) augmented with $\partial_y^2 \psi$, $\partial_t^2 \psi$, deduced at once from $\partial_x^2 \psi$ and $\partial_z^2 \psi$ respectively, we get for $A(z,t)$ the partial differential equation

$$(\partial_z^2 - \partial_t^2) A + 2i\omega\, (\partial_z A \partial_z \Omega - \partial_t A \partial_t \Omega) + i\omega A (\partial_x^2 \Omega + \partial_y^2 \Omega) - \omega_p^2 A = 0. \tag{A.10}$$

In this case also $\partial_z A = \partial_t A$ and since $\partial_x^2 \Omega = \partial_y^2 \Omega = -2D^{-1}$, this equation reduces to

$$4i\omega \partial_z A + 4i\omega\, D^{-1} A + \omega_p^2 A = 0 \tag{A.11}$$

similar to (A.7) except for a coefficient 4 instead of 2 in the second term and the solution of (A.11) is

$$A(z,t) = (t + z + ia)^{-1} \exp[-i\omega_p^2 (t + z + ia)/\omega]. \tag{A.12}$$

Note added in proof

After the completion of this work we became aware that dielectrics with memory are discussed in: A.C. Eringen, G.A. Maugen, *Electrodynamics of Continua* (Springer, New York, 1990), Chap. 13.

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